

Non-deterministic Chaos

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Non-deterministic chaos is a new dynamical paradigm where a non-deterministic system is influenced by random perturbations to produce the appearance of complexity. The non-determinism is envisioned to occur only at a single point in phase space, where many trajectories intersect. In the presence of external random perturbations (noise), whenever the phase space trajectory approaches the singularity, it will jump in an unpredictable way to a different solution. This behavior, while similar in appearance to deterministic chaos, has rather different implications for prediction and control.

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INTRODUCTION

It is often assumed that classical physical systems are *deterministic* [1], meaning that the forward time evolution is uniquely determined by the current state of the system. Correspondingly, determinism also implies that the system's current state uniquely determines its past behavior. These qualities follow directly from the *Existence and Uniqueness Theorem* of differential equations [2], which requires that the equations of motion of the system are Lipschitz continuous. However, this is nothing in classical mechanics that *requires* Lipschitz continuity. Indeed, in the case of a cracking whip, the physical solutions imply violation of the Lipschitz condition [3]. A similar effect is seen in seismic waves as they approach the surface of the Earth [4]. If Nature is not required to be Lipschitzian, we must ask if it required to be deterministic. Some types of non-determinism in classical systems have been explored previously [5,6]. In this work, we examine a somewhat different flavor of non-determinism, one which implies the possible existence of a new dynamical behavior: *non-deterministic chaos*.

THE NON-DETERMINISTIC HARMONIC OSCILLATOR (NDHO)

We begin with the simple harmonic oscillator (SHO), described by the equations

$$\begin{aligned}\frac{d}{dt}x &= y \\ \frac{d}{dt}y &= -x.\end{aligned}\tag{1}$$

Solutions of eqs. 1 are circles in the (x, y) phase plane. The SHO is a deterministic system, so that every point (x, y) belongs to a unique solution described by a circle with a particular radius $r = \sqrt{x^2 + y^2}$.

Suppose, now, that we apply the following non-linear coordinate transformation to the SHO phase space:

$$x \rightarrow x - r = x - \sqrt{x^2 + y^2}.\tag{2}$$

This translates all points on a circle of radius r in the positive x -direction by an amount equal to r . A family of circles concentric about the origin in the original space will now share a common tangent point at the origin of the transformed space (see Figure 1).

The equation of a transformed circle in the new space is given by

$$(x - r)^2 + y^2 = r^2,\tag{3}$$

or, solving for r ,

$$r = \frac{1}{2x}(x^2 + y^2). \quad (4)$$

Using eqn. 4, we can easily apply the transformation to the SHO. The transformed SHO equations of motion in the new coordinate system are given by

$$\begin{aligned} \frac{d}{dt}x &= y \\ \frac{d}{dt}y &= \frac{y^2}{2x} - \frac{1}{2}x. \end{aligned} \quad (5)$$

From the above discussion, the solutions of eqs. 5 will be the family of transformed circles, all sharing a common tangent point at the origin. Such intersection of many phase space trajectories is not so unusual. An attracting fixed point, for example, is approached asymptotically for all initial conditions in its basin of attraction (i.e., solutions are unique for finite times). What *is* unusual about eqs. 5 is that the common point is intersected in finite time, and further is not a fixed point. This is easily seen by taking the limit of eqs. 5 along a solution of radius r :

$$\begin{aligned} \lim_{x,y \rightarrow 0} \frac{d}{dt}x &= 0, \\ \lim_{x,y \rightarrow 0} \frac{d}{dt}y &= r. \end{aligned} \quad (6)$$

Thus, the origin is a singularity of eqs. 5, where neither past nor future time evolution is uniquely determined. Henceforth, we shall refer to eqs. 5 as the *non-deterministic harmonic oscillator* (NDHO).

The NDHO provides the paradigm for the type of system we are examining: solutions of the equations of motion are a family of closed loops (“transients”) all sharing a common tangent point. From eqs. 6, the dynamics of the NDHO are not defined by the equations of motion alone (this is not a necessary condition for a system to be non-deterministic [7]). However, let us imagine we built an NDHO in a laboratory. How would it behave? We note that all *physical* systems are subject to external perturbations, or “noise”. While the physical state of our NDHO is far (in phase space) from the point $(0,0)$, external noise will have little effect, provided the average amplitude of the fluctuations is small compared to r for that trajectory. However, as the trajectory approaches the origin, noise plays a larger role. Solutions for all r converge together, ultimately intersecting at $(0,0)$. Thus, noise will cause the trajectory to jump between solutions of widely differing r in a random way.

What is the effect of this on our laboratory NDHO? Suppose we begin the system on a solution of radius r_1 . As we watch the system evolve forward in time, we will find that after it passes near the origin the trajectory has changed to a completely different solution of radius r_2 . Repeating the experiment with the same initial conditions, we find that the trajectory jumps to a completely different solution of radius r_3 , where $r_3 \neq r_2$. Were we to repeat this a large number of times, for different values of r_1 , we would find that the solution after the singularity is completely unrelated to the solution before. If the NDHO were allowed to run for several oscillations, a time series measurement of one variable would appear as a piecewise continuous sequence of oscillations with different amplitudes. Further, the sequence of amplitudes would be random and unpredictable. We term this behavior *non-deterministic chaos*, non-deterministic because its origin lies in the non-determinism at a non-Lipschitz singularity, and chaos because of the long term unpredictability of the dynamics.

A PHYSICALLY MOTIVATED EXAMPLE

The NDHO, while illustrative of the type of non-determinism we are examining, is also a somewhat contrived example. We now describe a non-deterministic system based on physical considerations. This system is a model of the behavior of neutron star magnetic fields. We describe it briefly; for a more detailed discussion, the reader is referred to [8].

The model envisions two oppositely charged spherical shells which are allowed to rotate differentially. The magnetization of one shell (\mathbf{M}_1) will interact with the magnetic field of the second shell (\mathbf{H}_2), as well as experience

non-electromagnetic (“mechanical”) interactions with the surrounding medium. The magnetic interactions include a term to induce precession of \mathbf{M}_1 about the instantaneous direction of \mathbf{H}_2 , and the Landau-Lifshitz magnetic damping, which tends to align \mathbf{M}_1 with the direction of \mathbf{H}_2 . The mechanical interaction is taken as a simple damping, proportional to the difference in angular velocities of the two spheres. Parameterizing the interactions, we obtain the following equations:

$$\frac{d}{dt}\mathbf{M}_1 = \bar{\gamma}(\mathbf{M}_1 \times \mathbf{H}_2) - \bar{\lambda}\left(\frac{\mathbf{M}_1 \cdot \mathbf{H}_2}{M_1^2}\mathbf{M}_1 - \mathbf{H}_2\right) - \bar{\eta} \cdot (\omega_1 - \omega_2). \quad (7)$$

Following the scaling procedure described in [8], and conserving angular momentum, we arrive at the following equation:

$$\frac{d}{d\tau}\mathbf{m} = -\mathbf{m} \times \hat{z} - \lambda \left(\frac{\mathbf{m} \cdot \hat{z}}{m^2} \mathbf{m} - \hat{z} \right) - \bar{\epsilon}(\mathbf{m} - \hat{z}), \quad (8)$$

where \mathbf{m} is the scaled magnetization, τ is the scaled time, λ is the scaled Landau damping parameter, and $\bar{\epsilon}$ is the scaled viscous damping parameter tensor.

Examination of eqn. 8 reveals axial symmetry about the z -axis. This prompts us to make the following transformation:

$$\begin{aligned} x &= \sqrt{m_x^2 + m_y^2} \\ z &= m_z \\ \phi &= \arctan \frac{m_y}{m_x} \end{aligned} \quad (9)$$

which implies

$$\begin{aligned} m_x &\rightarrow x \cos \phi \\ m_y &\rightarrow x \sin \phi \\ m_z &\rightarrow z \end{aligned} \quad (10)$$

Substituting the above transformations into eqn. 8, we obtain

$$\begin{aligned} \dot{x} &= \frac{\lambda x z}{x^2 + z^2} - \epsilon x \\ \dot{z} &= \frac{\lambda z^2}{x^2 + z^2} - \bar{\epsilon} z - (\lambda - \bar{\epsilon}) \end{aligned} \quad (11)$$

and

$$\dot{\phi} = 1 \quad (12)$$

where an overdot again represents differentiation with respect to the scaled time τ . The ϕ equation is trivial, simply representing a constant precession about the z -axis. Any interesting dynamical behavior must occur in eqs. 11.

Numerical integration of eqs. 11 for $\epsilon < \lambda$ yields the phase space plot in Figure 2, and the time series in Figure 3. Note the apparent intersection of trajectories at the origin, indicating possible non-determinism at that point. Indeed, it can be shown [7,9] that for the RHS of eqs. 11, the point $(0, 0)$ is a non-Lipschitz singularity. Given this, the non-deterministic nature of eqs. 11 at $(0, 0)$ is proved by the following argument (for more details the reader is referred to [7,9]):

1. The only fixed point of eqs. 11 for $\epsilon < \lambda$ is at $(0, 1)$.
2. The sole fixed point is a saddle, and thus is not an attractor for a set of initial conditions of non-zero measure [10]. This also implies that no periodic orbits exist [10].
3. By the Poincaré-Bendixson Theorem, if the solutions of a two-degree of freedom dynamical system contain only Lipschitz points, then the only possible bounded asymptotic solutions are stationary or periodic. As neither possibility exists for eqs. 11, $(0, 0)$ must be contained by *all* bounded solutions, and therefore this point is non-deterministic.

NON-DETERMINISM AND PREDICTABILITY

Non-deterministic chaos has the property of being predictable for short times (between intersections of the singularity), yet completely unpredictable over long time periods. Long-term unpredictability is also one of the hallmarks of *deterministic* chaos, but it is here that the similarity ends. Aside from being described by deterministic equations, deterministic chaos is often characterized by exponential divergence of initially close solutions, and associated with an complex fractal structure, the strange attractor. Non-deterministic chaos derives its unpredictability from a more violent, but localized instability. Further, there exists no attractor, strange or otherwise, at least in the usual sense of the word.

Figures 4 illustrate the loss of information at the singularity due to the presence of external noise. A set of initial conditions is evolved via eqs. 11. Initially, we see the phase space stretching, similar to what occurs in deterministic chaos. However, once the singularity is encountered, points are scattered, and soon are randomly spread through a region of phase space (in this case, the region is enclosed by the homoclines of the saddle at (0,1) [7]). This behavior is in stark contrast to what one expects from deterministic chaos, where an initial volume is stretched and folded [11], spreading across the attractor in a smooth fashion. Dynamical measures such as the Lyapunov exponent are meaningless. The “attractor” exists only in a statistical sense, representing the probability density that a particular point in phase space will be visited. Formally, one could find this probability density by transforming eqs. 11 into a Brownian motion and numerically solving the forward Kolmogorov equation [12]. This approach is complicated by the existence of the singularity. Instead, we simply integrated eqs. 11 for 200 million time steps, and constructed the distribution from this. The results are shown in Figure 5.

Figure 5 gives a long-term, global statistical picture. However, the nature of non-deterministic chaos allows us to easily extract more useful statistical information. In particular, we shall utilize the fact that away from the singular point, the dynamics is quite well-behaved. Let us return to the NDHO, as its solutions are known analytically. The solutions of the NDHO may be parameterized by their radius. Solutions away from the singularity have essentially constant radii; the big jumps occur only near the singularity. Can we predict the probability that a circle of given r is chosen when the orbit leaves some neighborhood about the origin? Let us define this neighborhood as a disk of radius δ , and note that an orbit leaving this neighborhood does so with angle θ , which we take as measured from the y -axis. Now, assuming the external fluctuations to be isotropic, the probability density of picking a particular θ is constant, i.e.,

$$p(\theta)d\theta \propto d\theta. \quad (13)$$

Next, we note that everywhere except at the origin the Existence and Uniqueness Theorem applies to solutions of eqs. (5), thus each circle of radius r is associated with a unique θ , and we may write θ as a function of r . Substituting into $p(\theta)$, we find

$$p(r)dr \propto \frac{\partial\theta(r)}{\partial r}dr. \quad (14)$$

The probability of getting a circle between r and $r + \Delta r$ is simply

$$P(r, \Delta r) \propto \int_r^{r+\Delta r} \frac{\partial\theta(r)}{\partial r}dr = \theta(r + \Delta r) - \theta(r). \quad (15)$$

For the case at hand, we find

$$P(r, \Delta r) \propto \arccos \frac{\delta}{2(r + \Delta r)} - \arccos \frac{\delta}{2r}. \quad (16)$$

This approach (first described in [9]) is somewhat simplified. A rigorous derivation would account for the statistical properties of the noise, and derive $P(r, \Delta r)$ via stochastic calculus. The above does show, however, that the simple structure of the solutions of a non-deterministic system lends itself to the construction of statistical arguments. Further, with a judicious choice of δ , based on knowledge of the average amplitude of the fluctuations, the above procedure should yield a good approximation of the true distribution $P(r, \Delta r)$.

CONTROLLING NON-DETERMINISTIC CHAOS

The control of deterministic chaotic systems using small perturbations has been a subject of recent vigorous research [13]. The most popular method of controlling deterministic chaos involves the stabilization of (otherwise) unstable periodic orbits which are embedded in the chaotic motion. As there exist an infinity of orbits, a rich variety of behaviors may be extracted from the controlled deterministic chaotic system, allowing for flexibility and easy optimization of a system's behavior.

For a non-deterministic chaotic system, we have a similar situation. With a continuum of different solutions intersecting at a single point, we can easily effect control via an appropriate perturbation. Similar to the previous section, we simply examine how solutions leave a δ -neighborhood about the singularity. Again, away from the singular point the solution is well-defined. Suppose that each different solution may be parameterized by some quantity γ (in the case of the NDHO, this is the radius). A given solution, parameterized by γ_0 , will intersect the δ -neighborhood at a unique point (x_0, y_0) . From this, we may construct the angle $\theta(\gamma_0) = \arctan x_0/y_0$.

The angle $\theta(\gamma)$ we term the *control angle*, and the reason should be obvious. To keep the system on a solution with parameter γ_0 , we need only to wait until the trajectory approaches the origin, and then perturb it so that it leaves at angle $\theta(\gamma_0)$. This perturbation will be quite small, of the order of δ , with the size of δ being determined largely by the noise amplitude. We see that in a non-deterministic system, there is a continuum of possibilities available through small control perturbations. If a change in system behavior were required, it is easily and quickly effected by simply changing $\theta(\gamma)$. In fact, one could vary $\theta(\gamma)$ as a function of time to induce arbitrarily complex behavior.

As an example, we have applied this control algorithm to eqs. 11. To simulate the effect of noise, a small (10^{-4} of the integration stepsize) normally distributed random number was added at each integration step. The controlled signals for various values of θ are shown in Figure 6. Figure 7 shows the effect of noise for different values of δ .

DISCUSSION

The type of “non-determinism” described above should not be construed as implying stochasticity. Indeed, the behavior of both the NDHO and neutron star model are uniquely determined away from the singular point. It is at this point, and this point only, that the non-deterministic nature of the equations arises. In the presence of random fluctuations, which are ubiquitous (though perhaps small) in physical systems, the non-determinism, albeit it at a single point, becomes important. The resulting dynamics, which we have termed *non-deterministic chaos*, consist of a random sequence of “transient” oscillations.

This work does not represent the first suggestion that non-determinism exists in classical systems. The *terminal dynamics* described by M. Zak [5] utilizes a similar mechanism where multiple trajectories intersect at a common equilibrium point in finite time. Chen [6] has independently suggested the same behavior under the heading of *noise induced instability* (though non-determinism as such is never explicitly mentioned). For clarity, we shall henceforth refer to terminal dynamics as “Type I non-determinism”, and the dynamics described herein as “Type II non-determinism”. The primary difference between the two is that for Type I non-determinism, the singularity occurs at an equilibrium point of the equations of motion, while for Type II, the singularity is shared among a group of dynamic trajectories. The physical implications of this difference are yet to be explored. Other aspects of non-deterministic systems, especially in the presence of noise, have been explored by A. Hübner [14].

There is no principle in nature that precludes the existence of the types of systems we have described. Indeed, an analog VLSI circuit has been built which displays Type I non-determinism [15]. The neutron star model displays Type II non-determinism, and is based on perfectly reasonable physical assumptions. Work is in progress to build an electronic analog of this model. It has also been suggested that non-determinism may play an important role in biological systems [16]. However, the question arises as to the ubiquity of such systems. Is non-determinism a generic property of Nature, or have we simply stumbled upon a few pathological examples? We cannot at this time answer that question.

However, we feel further investigation is warranted. In particular, application of several standard measures (power spectrum, Lyapunov exponent, etc.) to a time series generated by eqs. 11 would lead one to believe that one examining

an instance of *deterministic* chaos [7]. As we have seen, though, issues of prediction and control would be addressed much differently for a non-deterministic chaotic system. Indeed, Crutchfield has shown that in the context of model building, assuming determinism when the underlying process is non-deterministic leads to undue complexity in the model [17]. It would seem reasonable to search for non-deterministic chaos in apparently complex systems, especially in cases where traditional analysis tools (which, again, assume determinism) have failed. In a forthcoming paper, we will address the problem of detecting non-determinism in observed data.

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FIG. 1. Some examples of circular orbits of different radii, all sharing a common point at the origin.

FIG. 2. Phase plot of solutions in the neutron star model for $\lambda = 1.0, \epsilon = 0.6, \bar{\epsilon} = 0.7$.

FIG. 3. Time series $z(t)$ vs. t for the neutron star model with parameter values $\lambda = 1.0, \epsilon = 0.6, \bar{\epsilon} = 0.7$.

FIG. 4. Loss of information due to the singularity. a) 10,000 initial points are arranged in a 100×100 square. b) Initial evolution. c) When the singularity is encountered, initially close points are scattered randomly. d) All information about the initial conditions has been lost. The only information carried by the system is in the density of trajectories.

FIG. 5. Probability of finding the system in a particular region of phase space. The distribution was found by integrating the equations for 200 million steps and totalling the amount of time spent in each region.

FIG. 6. Examples of output from the neutron star model when the control algorithm is applied. Signals are shown for $\theta(\gamma) =$ a) 0.005, b) 0.03, c) 0.08, and d) 0.2.

FIG. 7. The control algorithm begins to break down if δ is chosen to be comparable to the noise level. Signals are shown for $\delta =$ a) $10^4\sigma$, b) $10^3\sigma$, c) $10^2\sigma$, and d) 10σ , where σ is the RMS of the noise.